Q-conditional symmetry
of (1+2)-dimensional nonlinear heat equations

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All possible Q-conditional (non-classical) symmetry operators of (1+2)-dimensional nonlinear heat equations are classified with respect to the corresponding equivalence group and the corresponding set of admissible transformations. Reductions with respect to pure Q-conditional symmetry operators are carried out.

$$H(u)u_0 + \triangle u = F(u), \tag{1}$$

where $u = u(x) \in \mathbb{R}^1, x = (x_0, \vec{x}), \vec{x} \in \mathbb{R}^2, H(u)$ and $F(u)$ are arbitrary smooth functions. Any equation of form (1)) is reduced to more often usable from

$$u_0 + \nabla(g(u)\nabla u) = f(u), \tag{2}$$

by the transformation $u \to \int H(u)du$, where the functions f and g are expressed in terms of F and H.

Theorem 1. *Point equivalence group $G^\sim$ of class* (1) *is formed by the transformations*

$$
\begin{aligned}
&x_0 \to \beta_0 x_0 + \alpha_0, \qquad x_a \to \beta_1 x_a + \gamma_{ab} x_b + \alpha_a, \qquad u \to \beta_2 u + \alpha_3, \\
&H \to \beta_0 \beta_1^{-2} H, \qquad F \to \beta_2 \beta_1^{-2} F,
\end{aligned}
\tag{3}
$$

where $\beta_0, \beta_2 \neq 0$, $\beta_0 > 0$, $\beta_1 > 0$, α_l, γ_{ab} are constant, $(\gamma_{ab}) \in O(2)$, $a, b = 1, 2$, $l = \overline{0,3}$.

Theorem 2. *A complete list of $G^\sim$-inequivalent cases of extension of Lie invariance algebra of equations* (1) *are exhausted by ones adduced in Table 1.*

Table 1

№	$H(u)$	$F(u)$	Lie Invariance Algebra	Remarks
1	$\forall$	$\forall$	$A = \langle \partial_0,\ \partial_a,\ J_{12} = x_1\partial_2 - x_2\partial_1 \rangle$	
2	$\forall$	0	$A + \langle D = 2x_0\partial_0 + x_a\partial_a \rangle$	
3	e^{ku}	λe^{mu}	$A + \langle D_1 = 2(m-k)x_0\partial_0 + mx_a\partial_a - 2\partial_u \rangle$	$m \neq k$
3a	e^u	$\lambda_0 e^u + \lambda$	$A + \langle D_4 = e^{\lambda_0 x_0}(\partial_0 + \lambda_0\partial_u) \rangle$	$\lambda \neq 0$
4	u^k	λu^m	$A + \langle D_2 = 2(m-k-1)x_0\partial_0$ $+(m-1)x_a\partial_a - 2u\partial_u \rangle$	$(k,m) \neq (0,0),$ $m \neq k+1$
4a	u^k	$\lambda_0 u^{k+1} + \lambda u$	$A + \langle D_5 = e^{-\lambda_0 k x_0}(\partial_0 - \lambda_0 u\partial_u) \rangle$	$\lambda \neq 0$
5	1	$\lambda u \ln u$	$A + \langle e^{\lambda x_0}(\partial_a + \frac{\lambda}{2}x_a u\partial_u),\ e^{\lambda x_0}u\partial_u \rangle$	
6	u^k	0	$A + \langle D,\ D_3 = kx_a\partial_a - 2u\partial_u \rangle$	$k \neq 0$
6a	u^k	$\lambda_0 u^{k+1}$	$A + \langle D_3,\ D_5 \rangle$	$k \neq 0$
7	e^u	0	$\langle \partial_0,\ x_0\partial_0 + \partial_u,\ \xi^a(\vec{x})\partial_a - 2\xi_1^1\partial_u \rangle$	$\xi_2^1 + \xi_1^2 = 0,$ $\xi_1^1 = \xi_2^2$
7a	e^u	$\lambda_0 e^u$	$A + \langle D_4,\ \xi^a(\vec{x})\partial_a - 2\xi_1^1\partial_u \rangle$	$\xi_2^1 + \xi_1^2 = 0,$ $\xi_1^1 = \xi_2^2$

Here $\lambda \neq 0, m, k$ are constant, $\lambda \in \{-1; 1\}$ mod $G^\sim$. In Case 3 $k \in \{0; 1\}$ mod $G^\sim$ та $m \in \{0; 1\}$ mod $G^\sim$. In Cases 1 and 2 adduced algebras are maximal if the corresponding equation is inequivalent to ones of Cases 3–7.

Theorem 3. *Any Q-conditional symmetry operator of a nonlinear heat equation of form (1) is either equivalent to a Lie symmetry operator of this equation or equivalent with respect to $G^\sim$ and additional transformations to a one from operators adduced in Table 2, where λ, λ_0, λ_1, λ_2 are arbitrary constants.*

Table 2

N	$H(u)$	$F(u)$	Operators	Remarks
1	$\forall$	$(\lambda_1 u + \lambda_2)[H + \lambda_0]$	$Q = \partial_0 + (\lambda_1 u + \lambda_2)\partial_u$	$(\lambda_1, \lambda_2) \neq (0,0)$
2	u	$\lambda_2 u^2 + \lambda_1 u + \lambda_0$	$Q = \partial_0 + [\lambda_2 u - b(\vec{x})]\partial_u$	$\triangle b = b^2 + \lambda_1 b + \lambda_2\lambda_0$
3	1	$\lambda u \ln u,\ \lambda \neq 0$	$Q = \partial_1 + a(x_0, x_1)u\partial_u$	$a_0 + a_{11} = -2aa_1 + \lambda a$

Let us use the operators adduced in Table 2 for reduction of the corresponding equations of form (1) to differential equations with to independent variables. Each from these cases has interesting feathers and so was considered separately.

1. For arbitrary smooth function $H = H(u)$ and arbitrary constants $\lambda_0, \lambda_1, \lambda_2, (\lambda_1, \lambda_2) \neq (0,0)$, the equations

$$Hu_0 + \triangle u = (\lambda_1 u + \lambda_2)(H + \lambda_0) \tag{4}$$

is Q-conditionally invariant with respect to $Q = \partial_0 + (\lambda_1 u + \lambda_2)\partial_u$. Depending on relations between the coefficients λ_1 and λ_2 we have two inequivalent cases of construction of ansatzes:

    1. $\lambda_1 \neq 0$, then $\lambda_1 = 1$, $\lambda_2 = 0$ mod $G^\sim$: $u = e^{x_0}\varphi(\vec{x})$;

2. $\lambda_1 = 0$, $\lambda_2 \neq 0$, then $\lambda_2 = 1 \bmod G^\sim$: $u = x_0 + \varphi(\vec{x})$.

Here $\varphi = \varphi(\vec{x})$ is the new unknown equation. After substituting the ansatzes into equation (4), in both cases we obtain the reduced equation $\triangle \varphi = \lambda_0$.

Therefore, we can summarize for this case.

— This is the unique case for equation (1) when non-trivial Q-conditional symmetries are for the subclass of equations which are parameterized with an arbitrary function.

— It generalizes stationary ($\lambda_1 = \lambda_2 = 0$) solutions which are Lie invariant ones.

— Non-linear equation (4) for an arbitrary value of the parameter-function H is reduced to the linear equation which can be transformed by the point transformation $\tilde{\varphi} = \varphi - \lambda_0 x_1^2/2$ to the two-dimensional Laplace equation $\triangle \tilde{\varphi} = 0$.